Exam-3 Solutions, Math 10560

1. Find the sum of the following series

$$\sum_{n=1}^{\infty}\left[\frac{\ln(n+1)}{n+2}-\frac{\ln(n+2)}{n+3}\right]$$

.

Solution: First note that this series is a telescoping series. Let S_n be the nth partial sum of the series.

$$S_1 = \frac{\ln 2}{3} - \frac{\ln 3}{4}$$

$$S_2 = \frac{\ln 2}{3} - \cancel{\frac{\ln 3}{4}} + \cancel{\frac{\ln 3}{4}} - \frac{\ln 4}{5} = \frac{\ln 2}{3} - \frac{\ln 4}{5}$$

$$S_3 = \frac{\ln 2}{3} - \cancel{\frac{\ln 3}{4}} + \cancel{\frac{\ln 3}{4}} - \cancel{\frac{\ln 4}{5}} + \cancel{\frac{\ln 4}{5}} - \frac{\ln 5}{6} = \frac{\ln 2}{3} - \frac{\ln 5}{6}$$

$$S_n = \frac{\ln 2}{3} - \cancel{\frac{\ln 3}{4}} + \cancel{\frac{\ln 3}{4}} - \cancel{\frac{\ln 4}{5}} + \cancel{\frac{\ln 4}{5}} + \cdots - \cancel{\frac{\ln(n+1)}{n+2}} + \cancel{\frac{\ln(n+1)}{n+2}} - \frac{\ln(n+2)}{n+3}$$

This gives $S_n = \frac{\ln 2}{3} - \frac{\ln(n+2)}{n+3}$. Now,

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{\ln 2}{3} - \frac{\ln(n+2)}{n+3} \\
&= \frac{\ln 2}{3} - \lim_{n\to\infty} \frac{\ln(n+2)}{n+3} \\
&= \frac{\ln 2}{3} - \lim_{x\to\infty} \frac{\ln(x+2)}{x+3} \quad \text{[where } x \in \mathbb{R}\text{, i.e. } x \text{ is a real number]} \\
&= \frac{\ln 2}{3} - \lim_{x\to\infty} \frac{1}{x+2} \quad \text{[Using L'Hospital]} \\
&= \frac{\ln 2}{3} - 0 \\
&= \frac{\ln 2}{3}
\end{aligned}$$

This gives us

$$\sum_{n=1}^{\infty}\left[\frac{\ln(n+1)}{n+2}-\frac{\ln(n+2)}{n+3}\right] = \frac{\ln 2}{3}.$$

2. Use the comparison test or limit comparison test to determine which of the following series are convergent:

$$(I)\quad \sum_{n=2}^{\infty} \frac{\sin^2(n)+1}{2\sqrt{n}} \qquad (II)\quad \sum_{n=2}^{\infty} \frac{n^2+2n+1}{n^4+2n^2+1} \qquad (III)\quad \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

Solution: The answer is only (II) and (III) converge.

We first consider $\sum_{n=2}^{\infty} \frac{\sin^2(n)+1}{2\sqrt{n}}$. Using $\sin^2(n)+1 \geq 1$, we get $\frac{\sin^2(n)+1}{2\sqrt{n}} \geq \frac{1}{2\sqrt{n}}$. The series $\sum_{n=2}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-test. Now comparison test implies that $\sum_{n=2}^{\infty} \frac{\sin^2(n)+1}{2\sqrt{n}}$ diverges.

Next, we investigate $\sum_{n=2}^{\infty} \frac{n^2+2n+1}{n^4+2n^2+1}$. It looks like we want to use comparison test. But the problem is that it is not that clear to come up with a nicer series you want to compare with. Therefore we go for limit comparison test. Let $a_n = \frac{n^2+2n+1}{n^4+2n^2+1}$. We need b_n which is simpler. We find b_n by looking at the dominating factors in a_n which are n^2 in the numerator and n^4 in the denominator. Hence, we take $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$. Now,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{n^4+2n^2+1} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{n^4+2n^2+1} \\
&= \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{n^4+2n^2+1} \cdot \frac{1/n^4}{1/n^4} \\
&= \lim_{n\to\infty} \frac{1+2/n+1/n^2}{1+2/n^2+1/n^4} \\
&= 1
\end{aligned}$$

We know that the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-test. Limit Comparison Test implies that $\sum_{n=2}^{\infty} \frac{n^2+2n+1}{n^4+2n^2+1}$ converges too.

Finally, note that $\frac{1}{n2^n} \leq \frac{1}{2^n}$. The series $\sum_{1}^{\infty} \frac{1}{2^n}$ is a geometric series with common ratio $\frac{1}{2}$ and hence converges. Now by comparison test implies $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.

3. Consider the following series

$$(I) \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad\qquad (II) \quad \sum_{n=2}^{\infty} \frac{n}{\ln(n^2)} \qquad\qquad (III) \quad \sum_{n=1}^{\infty} \frac{3^{n+1}}{2(n!)}$$

Which of the following statements is true?

(a) Only (I) and (III) converge.
(b) Only(III) converges.
(c) Only (I) and (II) converge.
(d)All three converge.
(e) All three diverge.

Solution: Here the correct answer is only (I) and (III) converge.

Let $b_n = \frac{1}{\sqrt{n}}$. Then $b_n > 0$, b_n is decreasing and $\lim_{n\to\infty} b_n = 0$. Hence by the alternating series test the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

Note that $\frac{n}{\ln(n^2)} = \frac{n}{2\ln(n)} \geq \frac{1}{2}$ for all $n \geq 2$, because $\ln(n) \leq n$ for all $n > 0$. In other words, all of the terms in the sequence$\{\frac{n}{\ln(n^2)}\}_{n=2}^{\infty}$ are greater than $\frac{1}{2}$. This means that $\lim_{n\to\infty} \frac{n}{\ln(n^2)} \neq 0$. By divergence test, $\sum_{n=2}^{\infty} \frac{n}{\ln(n^2)}$ diverges.

For $\sum_{n=1}^{\infty} \frac{3^{n+1}}{2(n!)}$ we will use ratio test. Let $a_n = \frac{3^{n+1}}{2(n!)}$, then $a_{n+1} = \frac{3^{n+2}}{2(n+1)!}$. Using the fact that $(n+1)! = (n+1)n!$ we get $\frac{|a_{n+1}|}{|a_n|} = \frac{3}{n+1}$. Now, $\lim_{n\to\infty} \frac{3}{n+1} = 0$ which is less than 1. Applying ratio test we see that $\sum_{n=1}^{\infty} \frac{3^{n+1}}{2(n!)}$ is convergent.

4. Consider the following series

$$(I) \quad \sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n} \qquad\qquad (II) \quad \sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{2^n+1}\right)^n.$$

Which of the following statements is true?

(a) They both diverge.
(b) They both converge.
(c) (I) converges and (II) diverges.
(d) (I) diverges and (II) converges.
(e) Deciding whether these series converge or diverge is beyond the scope of the methods taught in this course.

Solution: The correct answer is they both diverge.

For $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n}$ we will use ratio test. Let $a_n = \frac{(n+1)!}{n^2 \cdot e^n}$, then $a_{n+1} = \frac{(n+2)!}{(n+1)^2 \cdot e^{n+1}}$. Now,

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{(n+2)!}{(n+1)^2 \cdot e^{n+1}} \cdot \frac{n^2 \cdot e^n}{(n+1)!} \\ &= \frac{n+2}{e} \cdot \frac{n^2}{(n+1)^2} \end{aligned}$$

which gives $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ which is greater than 1. Now by ratio test the series $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n}$ diverges.

Let $a_n = \left(\frac{2^{n+1}}{2^n+1}\right)^n$. Then $\sqrt[n]{|a_n|} = \frac{2^{n+1}}{2^n+1} = 2 \cdot \frac{2^n}{2^n+1} = 2 \cdot \frac{1}{1+2^{-n}}$. This gives $\lim_{n\to\infty} \sqrt[n]{a_n} = 2$ which is bigger than 1. Now by root test $\sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{2^n+1}\right)^n$ diverges.

5. Which **one** of the following series converges conditionally?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{3^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^5+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{e^n+1}$

Solution: Correct answer is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges conditionally.

The sum converges by the Alternating Series Test since $b_n = \frac{1}{\sqrt{n+1}} > 0$, $b_{n+1} = \frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n+1}} = b_n$, for all n and $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$. However, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test with $p = \frac{1}{2}$.

Note that a, b, and e are absolutely convergent and d is divergent.

6. Find a power series representation for the function

$$\frac{x^2}{(1-x^3)^2}$$

in the interval $(-1, 1)$.
(Hint: Differentiation of power series may help).

(a) $\sum_{n=1}^{\infty} nx^{3n-1}$

(b) $\sum_{n=1}^{\infty} (-1)^n 3nx^{3n-1}$

(c) $\sum_{n=1}^{\infty} nx^{n-1}$

(d) $\sum_{n=1}^{\infty} \frac{x^{3n+1}}{3n+1}$

(e) $\sum_{n=1}^{\infty} x^{3n}$

Solution: Write $f(x) = \frac{x^2}{(1-x^3)^2}$ We note that

$$\frac{d}{dx}\left[\frac{1}{1-x^3}\right] = \frac{3x^2}{(1-x^3)^2} = 3f(x).$$

We use the well-known power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1.$$

Plugging in x^3 we obtain:

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n} \text{ for } -1 < x < 1.$$

We differentiate this power series in order to compute a power series for $f(x)$.

$$\begin{aligned} 3f(x) = \frac{d}{dx}\left[\frac{1}{1-x^3}\right] &= \frac{d}{dx}\sum_{n=0}^{\infty} x^{3n} \\ &= \sum_{n=0}^{\infty} (3n)x^{3n-1}. \end{aligned}$$

So,

$$f(x) = \sum_{n=0}^{\infty} n x^{3n-1}$$

in the interval $(-1, 1)$.

7. Use you knowledge of a well known power series to calculate the limit

$$\lim_{x \to 0} \frac{2\cos(x^2) - 2 + x^4}{x^8}$$

(a) $\frac{1}{12}$

(b) $\frac{2}{8!}$

(c) $\frac{1}{2}$

(d) 2

(e) The limit does not exist

Solution: We know the Taylor (Maclaurin) series expansion for $\cos(x)$ around $x = 0$ is given by $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Substituting in x^2 we obtain

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

Now we can evaluate the limit as follows:

$$\begin{aligned}
\lim_{x \to 0} \frac{2\cos(x^2) - 2 + x^4}{x^8} &= \lim_{x \to 0} \frac{2\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots\right) - 2 + x^4}{x^8} \\
&= \lim_{x \to 0} \frac{(\cancel{2} - \cancel{x^4} + \frac{2x^8}{4!} - \frac{2x^{12}}{6!} + \frac{2x^{16}}{8!} - \cdots) - \cancel{2} + \cancel{x^4}}{x^8} \\
&= \lim_{x \to 0} \frac{1}{12} - \frac{2x^4}{6!} - \frac{2x^8}{8!} + \cdots \\
&= \frac{1}{12} + 0 + 0 + \cdots \\
&= \frac{1}{12}.
\end{aligned}$$

8. Which of the following is the third Taylor polynomial of the function

$$f(x) = \sin\left(\frac{x}{2}\right) \text{ centered at } a = \pi?$$

(a) $1 - \dfrac{(x-\pi)^2}{4(2!)}$

(b) $1 - \dfrac{\left(x - \frac{\pi}{2}\right)^2}{4(2!)}$

(c) $(x-\pi) - \dfrac{(x-\pi)^3}{3!}$

(d) $1 - \dfrac{x^2}{4(2!)}$

(e) $(x-\pi) - \dfrac{(x-\pi)^3}{2}$

Solution: The answer is $1 - \dfrac{(x-\pi)^2}{4(2!)}$.

Let $P_3(x)$ denote the third Taylor polynomial centered at $a = \pi$ of the function $f(x) = \sin\left(\frac{x}{2}\right)$. Then,

$$P_3(x) = f^{(0)}(\pi) + \frac{f'(\pi)}{1!}(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3$$

Hence, to find $P_3(x)$ we need to find the values: $f^{(0)}(\pi) = f(\pi), f'(\pi), f''(\pi), f'''(\pi)$.

$$\begin{aligned}
f^{(0)}(x) &= f(x) = \sin\left(\frac{x}{2}\right) \Rightarrow f^{(0)}(\pi) = \sin\left(\frac{\pi}{2}\right) = 1 \\
f'(x) &= \left[\sin\left(\frac{x}{2}\right)\right]' = \frac{1}{2}\cos\left(\frac{x}{2}\right) \Rightarrow f'(\pi) = \frac{1}{2}\cos\left(\frac{\pi}{2}\right) = 0 \\
f''(x) &= \left[\frac{1}{2}\cos\left(\frac{x}{2}\right)\right]' = -\frac{1}{4}\sin\left(\frac{x}{2}\right) \Rightarrow f''(\pi) = -\frac{1}{4}\sin\left(\frac{\pi}{2}\right) = -\frac{1}{4} \\
f'''(x) &= \left[-\frac{1}{4}\sin\left(\frac{x}{2}\right)\right]' = -\frac{1}{8}\cos\left(\frac{x}{2}\right) \Rightarrow f'''(\pi) = -\frac{1}{8}\cos\left(\frac{\pi}{2}\right) = 0
\end{aligned}$$

Hence,

$$\begin{aligned}P_3(x) &= f^{(0)}(\pi) + \frac{f'(\pi)}{1!}(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3 \\ &= 1 + \frac{0}{1!}(x-\pi) + \frac{-\frac{1}{4}}{2!}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 \\ &= 1 - \frac{(x-\pi)^2}{4(2!)}\end{aligned}$$

9. Compute the radius of convergence, R, of the following power series

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$$

(a) $R = 2$
(b) $R = 5$
(c) $R = \infty$
(d) $R = 1/2$
(e) $R = 1$

Solution: The answer is $R = 2$.
We use the ratio test: First we compute $\left|\frac{a_{n+1}}{a_n}\right|$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{2^{n+1}(n+1)^2}}{\frac{x^n}{2^n n^2}}\right| = \left|\frac{x \cdot \cancel{x^n}}{2 \cdot \cancel{2^n}(n+1)^2} \cdot \frac{\cancel{2^n} n^2}{\cancel{x^n}}\right| = \left|x \cdot \frac{n^2}{2(n+1)^2}\right| = |x| \cdot \frac{n^2}{2(n+1)^2}$$

So,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x| \cdot \frac{n^2}{2(n+1)^2} = \frac{1}{2}|x|$$

To get convergence, the Ratio Test says that we need $L < 1$ and so:

$$L = \frac{1}{2}|x| < 1 \Rightarrow |x| < 2$$

Hence, the radius of convergence $R = 2$.

10. Which of the following gives a power series representation of the function

$$f(x) = e^{-\frac{x^2}{2}}$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{2^n (2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{2^n n!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution: The answer is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$.

Indeed, we know that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and so if we plug in $-\frac{x^2}{2}$ we obtain:

$$e^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{2})^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}.$$

11. Consider the series $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{(\ln n)^2}{n}$. Fill in the following blanks and be sure to **show your work**. In each case indicate which test you are using and show how it is applied.

- Is the series absolutely convergent? (**YES** or **NO**) ____________

Solution: Answer is **No**. Note that $\displaystyle\sum_{n=3}^{\infty} \left|(-1)^n \frac{(\ln n)^2}{n}\right| = \sum_{n=3}^{\infty} \frac{(\ln n)^2}{n}$. For $n \geq 3$ we know that $\ln n \geq 1$ which gives $\frac{(\ln n)^2}{n} \geq \frac{1}{n}$. Recall that the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges.

By the Comparison Test, $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln n)^2}{n}$ diverges as well.

- Is the series convergent? (**YES** or **NO**) ____________

Solution: This series is an Alternating series. There is a possibility that we can use Alternating Series Test.

Let $b_n = \frac{(\ln n)^2}{n}$. Then $b_n > 0$. Now,

$$\begin{aligned}\lim_{n\to\infty} \frac{(\ln n)^2}{n} &= \lim_{x\to\infty} \frac{(\ln x)^2}{x} \quad \text{[Where } x \text{ is a real number.]}\\ &= \lim_{x\to\infty} 2\ln x \cdot \frac{1}{x} \quad \text{[Using L'Hospital]}\\ &= 2\lim_{x\to\infty} \frac{1}{x} \quad \text{[Using L'Hospital]}\\ &= 0\end{aligned}$$

If we can show b_n is decreasing then we can use the Alternating Series Test. We will use Calculus I to see if b_n is decreasing. Let $f(x) = \frac{(\ln x)^2}{x}$. Then $f'(x) = \frac{2\ln x - (\ln x)^2}{x^2} = \frac{\ln x(2-\ln x)}{x^2}$. For large x we have $2 - \ln x < 0$. This shows that $f'(x) < 0$ for large x and hence $f(x)$ is decreasing for large x.

Now by Alternating Series Test, $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{(\ln n)^2}{n}$ converges.

12. (a) Give the Taylor series expansion for the antiderivative

$$F(x) = \int \cos\left(\sqrt{x}\right) dx$$

about 0 (McLaurin Series) where $F(0) = 0$.
Hint: Use your knowledge of a well known series.

Solution: We know that the Taylor series expansion for $\cos(x)$ around $x = 0$ is $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, which has radius of convergence $R = \infty$. Plugging in $\sqrt{x}$ we obtain $\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$ which is valid for all $x \geq 0$. Finally, we compute the indefinite integral

$$\begin{aligned}F(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^n dx\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{n+1}}{n+1} + C\end{aligned}$$

Plugging in $x = 0$ we obtain

$$F(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{0^{n+1}}{n+1} + C = 0 + C = C.$$

So, $C = 0$, and

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{n+1}}{n+1}.$$

(b) Use part (a) to find an expression for the definite integral

$$\int_0^1 \cos(\sqrt{x})\,dx$$

as a sum of an infinite series.

Solution: By the Fundamental Theorem of Calculus we know

$$\begin{aligned}\int_0^1 \cos(\sqrt{x})\,dx &= F(1) - F(0) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1^{n+1}}{n+1} - 0 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{n+1}\end{aligned}$$

(c) Use the alternating series estimation theorem to estimate the value of the above definite integral so that the error of estimation is less than $\frac{1}{100}$.
(you may write your answer as a sum of fractions).

Solution: The series in part (b) is of the form $\sum_{n=0}^{\infty}(-1)^n b_n$ with $b_n = \frac{1}{(2n)!(n+1)}$. We check that this series satisfies the conditions for the Alternating Series Estimation Theorem

i) $b_{n+1} = \frac{1}{(2(n+1))!(n+2)} \leq \frac{1}{(2n)!(n+1)} = b_n$ holds for all $n \geq 0$,

ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(2n)!(n+1)} = 0.$

Thus $|R_n| = |S - S_n| \leq b_{n+1}$. We need to find the value of n which makes

$b_{n+1} < \frac{1}{100}$. We compute:

$$\begin{aligned} b_0 &= 1 \\ b_1 &= \frac{1}{2!(2)} = \frac{1}{4} \\ b_2 &= \frac{1}{4!(3)} = \frac{1}{72} \\ b_3 &= \frac{1}{6!(4)} = \frac{1}{720 \cdot 4} < \frac{1}{100} \end{aligned}$$

So $E_2 = |S - S_2| \leq b_3 < \frac{1}{100}$. So S_2 gives approximation of the integral which is within $\frac{1}{100}$ of the actual value. Finally, we compute our estimate for the integral,

$$\begin{aligned} \int_0^1 \cos(\sqrt{x})dx \approx S_2 &= b_0 - b_1 + b_2 \\ &= 1 - \frac{1}{4} + \frac{1}{72} \\ &= \frac{55}{72}. \end{aligned}$$

13. Find the radius of convergence and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{5^n \sqrt{n}}.$$

Solution: We use the ratio test: First we compute $\left|\frac{a_{n+1}}{a_n}\right|$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(x+3)^{n+1}}{5^{n+1}\sqrt{n+1}}}{\frac{(-1)^n(x+3)^n}{5^n\sqrt{n}}}\right| = \left|\frac{(x+3)\cdot \cancel{(x+3)^n}}{5\cdot \cancel{5^n}\sqrt{n+1}} \cdot \frac{\cancel{5^n}\sqrt{n}}{\cancel{(x+3)^n}}\right| = \left|(x+3)\cdot \frac{\sqrt{n}}{5\sqrt{n+1}}\right|$$

So,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|(x+3)\cdot \frac{\sqrt{n}}{5\sqrt{n+1}}\right| = \frac{|x+3|}{5} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \frac{|x+3|}{5}$$

To get convergence, the Ratio Test says that we need $L < 1$ and so:

$$L = \frac{|x+3|}{5} < 1 \Rightarrow |x+3| < 5$$

Hence, the radius of convergence is $R = 5$. Now, to get the interval of convergence we have that it contains $(a-R, a+R)$ where $a = -3, R = 5$. That is, I.O.C contains the interval $(-8, 2)$. What is left to do is to check the end points to see if they belong to the interval of convergence.

$\underline{x = -8:}$ $$\sum_{n=1}^{\infty} \frac{(-1)^n(-8+3)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}\cancel{5^n}}{\cancel{5^n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

So, for $x = -8$ the series diverges since it is a p series with $p = \frac{1}{2} \leq 1$. Therefore, $x = -8$ is NOT in the interval of convergence.

$\underline{x = 2:}$ $$\sum_{n=1}^{\infty} \frac{(-1)^n(2+3)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n\cancel{5^n}}{\cancel{5^n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

So, for $x = 2$ the series converges by the Alternating Series Test. Therefore, $x = 2$ IS in the interval of convergence.

Hence, the interval of convergence is $(-8, 2]$.

Formula Sheet

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$